Amalgamated Worksheet # 5

Induction Workshop

May 5, 2013

1 Induction warmup

Problem 1: (Sum of the first n integers) Prove by induction that

$$1 + 2 + \cdots + n = \frac{1}{2}n(n+1).$$

Problem 2: (Geometric series) Prove by induction that

$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{(n+1)}}{1 - r},$$

where $r \neq 1$

Problem 3: Correction - show that $4^n + 5$ is divisible by 3. (The original Problem 3 said to show that $4^n - 5$ is divisible by 3, which is false.)

Problem 4: Show that $4^n + 15n - 1$ is divisible by 9 for all n.

Hint: use Problem 3

Problem 5: Prove that $n! > 2^n$ for all $n \geq 4$

Problem 6: Let a and b be two distinct integers and n any positive integer. Prove that $a^n - b^n$ is divisible by $(a - b)$.

Hint: Use the identity

$$(a^{n+1} - b^{n+1}) = \frac{1}{2}((a^n + b^n)(a - b) + (a^n - b^n)(a + b)).$$

2 Induction on steroids

The Peyam's way

Show by induction on n that the determinant of an upper-triangular $n \times n$ matrix A equals to the product of its diagonal entries.

Note: Use the Math 54-definition of determinants (in terms of cofactor expansions).

The Daniel's way

Preliminary: If a, b are natural numbers at least 1, then we say a divides b, and we write $a|b$, if there exists a third natural number d such that $b = da$. You may use the following facts:

- If $m|n$ and $m'|n'$, then $mm'|nn'$.
- If $(ab)|c$ then $a|c$.
- If p is prime, and $p|(mn)$ then $p|m$ or $p|n$.

You may also find useful the formulas

- **(Geometric series)** $\dfrac{x^r - 1}{x - 1} = x^{r-1} + x^{r-2} + \cdots + 1.$
- **(Difference of squares)** $(a^2 - b^2) = (a - b)(a + b).$

Problem 1: Prove that $3 \cdot 2^{n+2}$ divides $(5^{2^n} - 1)$ for all $n \geq 1$.

Problem 2: Let p be a prime, and let n, a, b be natural numbers at least 1. Prove that if $p^n|(ab)$, and $p \not| b$, then $p^n|a$.

Problem 3: Prove that 2^{n+2} divides $(7^{2^n-1}+7^{2^n-2}+\cdots+7+1)$ for $n \geq 1$. Suggestion: Try making this into a statement more like the one in Problem 1.

The Mohammad's way

Let A, B be self adjoint operators on a finite dimensional inner product space V such that $AB = BA$. Prove, by induction on $\dim V$, that there exists an orthonormal basis of V whose elements are eigenvectors for both A and B.

The Lisha's way

There are 20 people in a room, each with either a blue or red hat on their head. Each person can clearly see the hat on everyone else head, but cannot see their own. In fact 8 people have blue hats and 12 have red. The game is played in rounds, and in each round the people who know the color of their hats raise their hands. The game proceeds this way an exhausting number of rounds and no one raises their hands.

Then a child wanders into the room and exclaims: "a blue hat!" Eight rounds later all eight people with blue hats raise their hands.

1. Explain why this happened. i.e. show that if there were k blue hats all k people would raise their hands in the k-th round.

2. What do you expect to happen in the 9th round?

3. Everyone in the room already knew that there was at least one person had a blue hat, since everyone could see at least seven blue hats. Then why did the child's exclamation have any effect?

3 Determinants and other delicacies

(by Master Peyam Tabrizian)

Note: In the following problems, use the Math 54-definition of determinants (in terms of cofactor expansions).

Problem 1: If $\mathbb{F} = \mathbb{C}$, show that the determinant of a linear operator T equals to the product of its eigenvalues, including multiplicities[1]

Note: You may use the fact that the determinant of T equals to the determinant of $\mathcal{M}(T)$, which is independent of the basis of V that you choose for $\mathcal{M}(T)$.

Problem 2:

1. (a) Do there exist linear operators S and $T \in \mathcal{L}(V)$ such that $TS - ST = I$?

2. (b) Is there a 3×3 matrix A with $A^2 = -I$?

[1]The latter one is Axler's definition of the determinant. This problem shows that Axler's definition of the determinant and the Math 54-definition of the determinant are equivalent

Problem 3: Without using any integrals, calculate the volume of the ellipsoid

$$E = \left\{ (x, y, z) \in \mathbb{R}^3 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \right\}$$

Where a, b, c are nonnegative real numbers.